Meaning and Logic

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Abstract

Semiotics for semantics and soundness are shown reasonable.

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1. Definition

Notation 1.1 (Meta-variables) *An infinite supply of variables, V, are denoted by lower-case roman letters (*x, y, z, *. . .) and λ-terms, Λ, are denoted by upper-case roman letters (*M, N, O, *. . .).*

Definition 1.2 (Lambda Terms) *A λ*-term *is either a* variable*, an* abstraction*, or an* application*:*

$$\begin{array}{llll} x \in V & \Rightarrow & x & \in \Lambda \quad \textit{Variable} \\ x \in V, M \in \Lambda & \Rightarrow & (\lambda x.M) & \in \Lambda \quad \textit{Abstraction} \\ M, N \in \Lambda & \Rightarrow & (MN) & \in \Lambda \quad \textit{Application} \end{array}$$

Notation 1.3 (Bracketing) *Outermost brackets may be discarded,* $(M(NO)) \rightarrow M(NO)$*. Application associates to the left,* $(MN)O \rightarrow MNO$*, and abstraction associates to the right,* $\lambda x.(\lambda y.(\lambda z.M)) \rightarrow \lambda x.\lambda y.\lambda z.M$*. Consecutive abstractors can be abbreviated as one,* $\lambda x.\lambda y.\lambda z.M \rightarrow \lambda x\,y\,z.M$*.*

Definition 1.4 (Bound Variables) *In a λ-term, λx.M, the variable x is* bound *by the abstractor λ and is under the abstractor's* scope.

Definition 1.5 (Free Variables)

1. *A variable occuring in a λ-term that is not bound is* free*. The set of* free variables *is defined inductively:*

$$\begin{array}{rcl} FV(x) & = & \{x\} \\ FV(\lambda x.M) & = & FV(M) \setminus \{x\} \\ FV(MN) & = & FV(M) \cup FV(N) \end{array}$$

2. *M is* closed *or a* combinator *if* $FV(M) = \{\emptyset\}$.
3. $\Lambda^0 = \{M \in \Lambda \mid M \textit{is closed}\}$.
4. $\Lambda^0(\vec{x}) = \{M \in \Lambda \mid FV(M) \subseteq \{\vec{x}\}\}$.
5. *A* closure *of* $M \in \Lambda$ *is* $\lambda\vec{x}.M$*, where* $\{\vec{x}\} = FV(M)$.

Definition 1.6 (Subterms)

1. *M is a* subterm *of N* ($M \subset N$) *if* $M \in Sub(N)$ *where* $Sub(N)$*, the collection of subterms of N, is defined inductively:*

$$\begin{array}{rcl} Sub(x) & = & \{x\} \\ Sub(\lambda x.N_1) & = & Sub(N_1) \cup \{\lambda x.N_1\} \\ Sub(N_1 N_2) & = & Sub(N_1) \cup Sub(N_2) \cup \{N_1 N_2\}. \end{array}$$

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2. *A subterm may* occur *several times.*
3. *Let N_1, N_2 be subterm occurrences of M. Then N_1, N_2 are* disjoint *if N_1 and N_2 have no common symbol occurrences.*
4. *A subterm occurrence N of M is* active *if N occurs as $(NZ) \subset M$ for some Z; otherwise N is* passive.

Convention 1.7 (Variable Convention) *If $M_1, \ldots, M_n$ occur in a certain mathematical context, then in these terms all bound variables are chosen different from the free variables.*

Notation 1.8 (Syntactic Equality) *For any two λ-terms M and N, $M \equiv N$ denotes syntactic equality.*

Definition 1.9 (α-conversion, Church 1941) *The renaming of bound variables is called α-*conversion.

$$\lambda x.M \stackrel{\alpha}{\to} \lambda y.[y/x]M \quad \textit{provided that } y \textit{ does not occur in } M$$

Two terms that are the same up to renaming of variables are equivalent.

$$M \stackrel{\alpha}{\to} N \Rightarrow M \equiv N$$

De Bruijn terms, which label variables by position (distance from the closest left-hand abstractor) are a means of avoiding variable renaming issues Barendregt (1984).

Definition 1.10 (Equality) *Equality is an equivalence relation over λ-terms:*

$$\begin{array}{rcll} M & = & M & Reflexive \\ M = N & \Rightarrow & N = M & Symmetric \\ M = N, N = O & \Rightarrow & M = O & Transitive \end{array}$$

Definition 1.11 (Compatibility)

$$\begin{array}{rcl} M = M' & \Rightarrow & MZ = M'Z \\ M = M' & \Rightarrow & ZM = ZM' \\ M = M' & \Rightarrow & \lambda x.M = \lambda x.M' \end{array}$$

As a consequence, any (sub)term can be replaced by equal terms in any term context.

Notation 1.12 (Provability) *If $M = N$ is provable in the λ-calculus we can write:*

$$\lambda \vdash M = N$$

Definition 1.13 (Consistency)

1. *An* equation *is a formula of the form $M = N$ with $M, N \in \Lambda$; the equation is* closed *if $M, N \in \Lambda^0$.*
2. *Let $\mathfrak{F}$ be a formal theory with equations as formulas. Then $\mathfrak{F}$ is* consistent *($Con(\mathfrak{F})$) if $\mathfrak{F}$ does not prove every closed equation, in the opposite case $\mathfrak{F}$ is* inconsistent.
3. *If $\mathfrak{F}$ is a set of equations, then $\lambda + \mathfrak{F}$ is the theory obtained from λ by adding the equations of $\mathfrak{F}$ as axioms. $\mathfrak{F}$ is called* consistent *($Con(\mathfrak{F})$) if $Con(\lambda + \mathfrak{F})$.*

2. Substitution

Definition 2.1 (Substitution) *The expression* $[N/x]M$ *is the term* M *with all free occurrences of* x substituted *for by* N*:*

$$\begin{array}{rcl} [N/x]x & \to & N \\ [N/x]y & \to & y \\ [N/x](\lambda y.M) & \to & \lambda y.([N/x]M) \\ [N/x](M_1 M_2) & \to & ([N/x]M_1)([N/x]M_2) \end{array}$$

Lemma 2.2 (Substitution Lemma) *If* $x \not\equiv y$ *and* $x \notin FV(L)$*, then*

$$[L/y][N/x]M = [[L/y]N/x][L/y]M$$

Proof By induction on structure of M.

1. M is a variable:
   (a) $M \equiv x \Rightarrow$ both sides equal $[L/y]N$ since $x \not\equiv y$
   (b) $M \equiv y \Rightarrow$ both sides equal L, for $x \notin \mathrm{FV}(L) \Rightarrow [\ldots/x]L \equiv L$
   (c) $M \equiv z \not\equiv x, y \Rightarrow$ both sides equal z
2. $M \equiv \lambda z.M_1$: By the variable convention we may assume that $z \not\equiv x, y$ and z not free in N, L. Then by the induction hypothesis

$$\begin{array}{rcl} [L/y][N/x](\lambda z.M_1) & \equiv & \lambda z.[L/y][N/x]M_1 \\ & \equiv & \lambda z.[[L/y]N/x][L/y]M_1 \\ & \equiv & (\lambda z.M_1)[[L/y]N/x][L/y]. \end{array}$$

3. $M \equiv M_1 M_2$: Then by the induction hypothesis the statement follows.

□

Proposition 2.3 1. $M = M' \Rightarrow [N/x]M = [N/x]M'$
2. $N = N' \Rightarrow [N/x]M = [N'/x]M$
3. $M = M', N = N' \Rightarrow [N/x]M = [N'/x]M'$

Proof

1. By induction on the length of proof $M = M'$
   (a) $M = M'$ is an axiom $(\lambda y.A)B = [B/y]A$. Then

$$\begin{array}{rcl} [N/x]M & \equiv & (\lambda y.[N/x]A)[B/y]A \\ & = & [[N/x]B/y][N/x]A \\ & \equiv & [N/x][B/y]A \\ & \equiv & [N/x]M'. \end{array}$$

   by the substitution lemma 2.2. (By the variable convention $y \not\equiv x$ and $y \notin \mathrm{FV}(N)$).
   (b) $M = M'$ is an axiom because $M \equiv M'$. Then the result follows immediately.
   (c) $M = M'$ is $ZM_1 = ZM_1'$ and is a direct consequence of $M = M_1'$. Hence

$$\begin{array}{rcl} [N/x]M & \equiv & [N/x]Z[N/x]Z_1 \\ & = & [N/x]Z[N/x]M' \text{ by induction hypothesis} \\ & \equiv & [N/x]M'. \end{array}$$

   The other derivation rules are treated similarly.
2. Induction on the structure of M.
3. by 1 and 2

$$[N/x]M = [N/x]M' = [N'/x]M'.$$

□

Definition 2.4 (Contexts and Holes) 1. *A* context $C[\ ]$ *is a term with some holes in it. More formally:*

x is a context
$[\ ]$ is a context
if $C_1[\ ]$ and $C_2[\ ]$ are contexts, then so are $C_1[\ ]C_2[\ ]$ and $\lambda x.C_1[\ ]$.

2. *If $C[\ ]$ is a context and $M \in \Lambda$, then $C[M]$ denotes the result of placing M in the holes of $C[\ ]$. In this act free variables of M may become bound in $C[M]$.*

Proposition 2.5 *Let $C[\ ] \in \Lambda$. Then*

$$N = N' \Rightarrow C[N] = C[N']$$

Proof Induction on the structure of $C[\ ]$. □

Lemma 2.6 1. $\forall C[\ ], \forall \vec{x}, \exists F : \forall M \in \Lambda^0(\vec{x}), C[M] = F(\lambda \vec{x}.M)$.
2. $\forall C[\ ], \forall M, \exists \vec{x} : \exists F : C[M] = F(\lambda \vec{x}.M)$.

Proof

1. Induction on the structure of $C[\ ]$.
2. By 1.

□

Definition 2.7 (Vector Terms) 1. *Let $\vec{N} \equiv N_1, \ldots, N_m; \vec{x} \equiv x_1, \ldots, x_n$. Then $\vec{N}$* fits in *$\vec{x}$ if $m = n$ and the $\vec{x}$ do not occur in $FV(\vec{N})$.*
2. *Let $\vec{N} \equiv N_1, \ldots, N_m; \vec{L} \equiv L_1, \ldots, L_n$. Then*

$$\vec{N} = \vec{L} \text{ if } m = n \text{ and } N_i = L_i, \quad \forall i\ 1 \leq i \leq n.$$

Similarly $\vec{N} = \vec{L}$ is defined.
3. *Let $\vec{N}$ fit in $\vec{x} = x_1, \ldots, x_n$. Then*

$$[\vec{N}/\vec{x}]M \equiv [N_n/x_n] \cdots [N_1/x_1]M.$$

4. *Let $M \in \Lambda$. Sometimes we write $M \equiv M(\vec{x})$ to indicate substitution:*
If $M \equiv M(\vec{x})$ and $\vec{N}$ fits in $\vec{x}$, then $M(\vec{N}) \equiv [\vec{N}/\vec{x}]M$.

Proposition 2.8 *Let $\vec{N}$ fit in $\vec{x}$. Then*

$$M_1(\vec{x}) = M_2(\vec{x}) \wedge \vec{N}_1 = \vec{N}_2 \Rightarrow M_1(\vec{N}_1) = M_2(\vec{N}_2)$$

Proof By proposition 2.3 (3). □

Lemma 2.9 *Let $\vec{x} = x_1, \ldots, x_n$. Then $(\lambda \vec{x}.M)\vec{x} = M$.*

Proof If $n = 1$, then

$$(\lambda x_1.M)x_1 = [x_1/x_1]M \equiv M.$$

If $n = 2$, then

$$\begin{aligned} (\lambda \vec{x}.M)\vec{x} &\equiv ((\lambda x_1.(\lambda x_2.M))x_1)x_2 \\ &= (\lambda x_2.M)x_2 \quad \text{by the case } n = 1 \text{ for } \lambda x_2.M \\ &= M. \end{aligned}$$

The general case follows by induction on n. □

Corollary 2.10 (Combinatory Completeness) *Let $M \equiv M(\vec{x})$. Then:*

1. $\exists F : F\vec{x} = M(\vec{x})$.
2. $\exists F : \forall \vec{N}, F\vec{N} = M(\vec{N})$*, where $\vec{N}$ fits in $\vec{x}$.*
3. *In 1, 2 one can take $F \equiv \lambda \vec{x}.M$.*

Proof By lemma 2.9, using proposition 2.8. □

3. Named Terms

Definition 3.1 (μ-conversion) *λ-terms can be given names and the μ-*conversion *of a term involving these names is the term with the name replaced by the corresponding λ-term*

$$\mathbf{name} := M \Rightarrow (\mathbf{name}\ N) \stackrel{\mu}{\to} MN$$

Definition 3.2 (Standard Combinators)

$$\begin{array}{rcl} \mathbf{I} & := & \lambda x.x \\ \mathbf{K} & := & \lambda x\,y.x \\ \mathbf{S} & := & \lambda x\,y\,z.x\,z\,(y\,z) \end{array}$$

Corollary 3.3 $\forall M, N, L \in \Lambda$

1. $\mathbf{I}M = M$
2. $\mathbf{K}MN = M$
3. $\mathbf{S}MNL = ML(NL)$.

Proof By corollary 2.10. □

Theorem 3.4 $\forall M \in \Lambda, \mathbf{SK}M = \mathbf{I}$

Proof By repeated application of Lemma 4.2.

$$\begin{array}{rcl} \mathbf{SK}M & = & (\lambda x\,y\,z.x\,z(y\,z))(\lambda x\,y.x)M \\ & = & (\lambda y\,z.(\lambda x\,y.z)z(y\,z))M \\ & = & (\lambda y\,z.z)M \\ & = & \lambda z.z \\ & = & \mathbf{I}. \end{array}$$

□

4. Reduction

Definition 4.1 (β-reduction) *The principal axiom scheme is* β-reduction*:*

$$(\lambda x.M)N \xrightarrow{\beta} [N/x]M \quad \forall M, N \in \Lambda$$

Lemma 4.2

$$\lambda \vdash (\lambda x_1 \cdots x_n.M)X_1 \cdots X_n = [X_n/x_n] \cdots [X_1/x_1]M$$

Proof From axiom (β-reduction) we have

$$(\lambda x_1.M)X_1 = [X_1/x_1]M$$

by induction on n the result follows. □

Theorem 4.3 (FixedPoint Theorem)
1. $\forall F \in \Lambda, \exists X \in \Lambda : FX = X$
2. *There is a fixed point combinator*

$$\mathbf{Y} := \lambda f.(\lambda x.f(x\,x))(\lambda x.f(x\,x))$$

such that

$$\forall F \in \Lambda, F(\mathbf{Y}F) = \mathbf{Y}F$$

Proof
1. Define $W := \lambda x.F(x\,x)$ and $X := WW$. Then

$$X = WW = (\lambda x.F(x\,x))W = F(WW) = X$$

2. By proof of 1

□

Definition 4.4 (β-Normal Form) *Let $M \in \Lambda$.*
1. *M is a* β-normal form *(β-nf or nf) if M has no subterm $(\lambda x.P)Q$.*
2. *M has a β-normal form if $\exists N : N = M$ and N is a β-normal form.*
3. *If M is a nf, it is also said that M is* in *nf.*

Definition 4.5 (η-Reduction) *Redundant abstractions can be removed with* η-reduction

$$\lambda x.Mx \xrightarrow{\eta} M \quad x \notin FV(M)$$

The theory λ extended with η-reduction is called λ_η.

Definition 4.6 ($\beta\eta$-Normal Form)
1. *M is a $\beta\eta$-normal form if M has no subterm $(\lambda x.P)Q$ or $(\lambda x.R\,x)$ with $x \notin FV(R)$.*
2. *M has a $\beta\eta$-normal form if*

$$\exists N : \lambda_\eta \vdash M = N \wedge N \text{is a } \beta\eta\text{-normal form}$$

Definition 4.7 (Head Normal Form)
1. *M is a* head normal form *(hnf) if M has the form $M \equiv \lambda \vec{x}.y\,\vec{N}$.*
2. *M has a hnf if $\exists N : M = N$ and N is a hnf.*

Definition 4.8 (Weak Head Normal Form)
1. *M is a* weak head normal form *(whnf) if M has the form $M \equiv \lambda \vec{x}.\vec{N}$.*
2. *M has a whnf if $\exists N : M = N$ and N is a whnf.*

These head normal forms provide intermediate points towards unique normal forms. Weak head normal form is useful as it provides for a *lazy* evaluation strategy that does not evaluate terms until necessary.

Theorem 4.9

$$\forall M \in \Lambda^0(\vec{x}, y), \exists F : \forall \vec{A}, F\vec{A} = M(\vec{A}, F).$$

Proof As a result of combinatory completeness and the fixedpoint theorem. □

Definition 4.10 1. *A binary relation* $\mathbf{R}$ *on* Λ *is* compatible *(with the operations) if*

$$\begin{array}{rcl} (M, M') \in \mathbf{R} & \Rightarrow & (ZM, ZM') \in \mathbf{R} \\ & \wedge & (MZ, M'Z) \in \mathbf{R} \\ & \wedge & (\lambda x.M, \lambda x.M') \in \mathbf{R}. \end{array}$$

2. *An* equality *(or* congruence*)* relation *on* Λ *is a compatible equivalence relation.*
3. *A* reduction relation *on* Λ *is one which is compatible, reflexive, and transitive.*

Definition 4.11 1. *A* notion of reduction *on* Λ *is just a binary relation* $\mathbf{R}$ *on* Λ.
2. *If* $\mathbf{R_1}, \mathbf{R_2}$ *are notions of reduction, then* $\mathbf{R_1 R_2}$ *is* $\mathbf{R_1} \cup \mathbf{R_2}$.

Definition 4.12 $\beta = \{((\lambda x.M)N, [N/x]M) | \; M, N \in \Lambda\}$.

Definition 4.13 *If* $\mathbf{R}$ *is a binary relation on a set X, then the* reflexive closure *of the relation is the least relation extending it that is reflexive. The* transitive closure *and the* compatible closure *are defined similarly.*

Definition 4.14 (Reduction Relations) *Let* $\mathbf{R}$ *be a notion of reduction on* Λ.

1. *Then* $\mathbf{R}$ *induces the binary relations:*
   (a) $\stackrel{\mathbf{R}}{\to}$ *one step* $\mathbf{R}$*-reduction*
   (b) $\stackrel{\mathbf{R}}{\twoheadrightarrow}$ $\mathbf{R}$*-reduction*
   (c) $\stackrel{\mathbf{R}}{=}$ $\mathbf{R}$*-equality (or convertibility)*
   inductively defined:
   (a) $\stackrel{\mathbf{R}}{\to}$ *is the compatible closure of* $\mathbf{R}$*:*
      i. $(M, N) \in \mathbf{R} \Rightarrow M \stackrel{\mathbf{R}}{\to} N$
      ii. $M \stackrel{\mathbf{R}}{\to} N \Rightarrow ZM \stackrel{\mathbf{R}}{\to} ZN$
      iii. $M \stackrel{\mathbf{R}}{\to} N \Rightarrow MZ \stackrel{\mathbf{R}}{\to} NZ$
      iv. $M \stackrel{\mathbf{R}}{\to} N \Rightarrow \lambda x.M \stackrel{\mathbf{R}}{\to} \lambda x.N$
   (b) $\stackrel{\mathbf{R}}{\twoheadrightarrow}$ *is the reflexive, transitive closure of* $\mathbf{R}$*:*
      i. $M \stackrel{\mathbf{R}}{\to} N \Rightarrow M \stackrel{\mathbf{R}}{\twoheadrightarrow} N$
      ii. $M \stackrel{\mathbf{R}}{\twoheadrightarrow} M$
      iii. $M \stackrel{\mathbf{R}}{\twoheadrightarrow} N, N \stackrel{\mathbf{R}}{\twoheadrightarrow} L \Rightarrow M \stackrel{\mathbf{R}}{\twoheadrightarrow} L$
   (c) $\stackrel{\mathbf{R}}{=}$ *is the equivalence relation generated by* $\stackrel{\mathbf{R}}{\to}$*:*
      i. $M \stackrel{\mathbf{R}}{\twoheadrightarrow} N \Rightarrow M \stackrel{\mathbf{R}}{=} N$
      ii. $M \stackrel{\mathbf{R}}{=} N \Rightarrow N \stackrel{\mathbf{R}}{=} M$
      iii. $M \stackrel{\mathbf{R}}{=} N, N \stackrel{\mathbf{R}}{=} L \Rightarrow M \stackrel{\mathbf{R}}{=} L$
2. *The basic relations derived from* $\mathbf{R}$ *are pronounced as follows:*
   (a) $M \stackrel{\mathbf{R}}{\twoheadrightarrow} N$ *: M* R-reduces to *N or N is an* R-reduct *of M*
   (b) $M \stackrel{\mathbf{R}}{\to} N$ *: M* R-reduces to *N* in one step
   (c) $M \stackrel{\mathbf{R}}{=} N$ *: M is* R-convertible to *N*

Lemma 4.15 *The relations* $\stackrel{\mathbf{R}}{\to}$, $\stackrel{\mathbf{R}}{\twoheadrightarrow}$, *and* $\stackrel{\mathbf{R}}{=}$ *are all compatible. Therefore* $\stackrel{\mathbf{R}}{\twoheadrightarrow}$ *is a reduction relation and* $\stackrel{\mathbf{R}}{=}$ *is an equality relation.*

Proof For $\stackrel{\mathbf{R}}{\to}$ this is immediate. For $\stackrel{\mathbf{R}}{\twoheadrightarrow}$ and $\stackrel{\mathbf{R}}{=}$ it follows by induction on the generation of these relations. This is a common form of proof. For $\stackrel{\mathbf{R}}{\twoheadrightarrow}$:

1. $M \stackrel{\mathbf{R}}{\twoheadrightarrow} N$ because $M \stackrel{\mathbf{R}}{\to} N$. Then by the result for $\stackrel{\mathbf{R}}{\to}$ one has $C[M] \stackrel{\mathbf{R}}{\to} C[N]$ and hence $C[M] \stackrel{\mathbf{R}}{\twoheadrightarrow} C[N]$.
2. $M \stackrel{\mathbf{R}}{\twoheadrightarrow} N$ because $M \equiv N$. Then trivially $C[M] \stackrel{\mathbf{R}}{\twoheadrightarrow} C[N]$.
3. $M \stackrel{\mathbf{R}}{\twoheadrightarrow} N$ is a direct consequence of $M \stackrel{\mathbf{R}}{\twoheadrightarrow} L$ and $L \stackrel{\mathbf{R}}{\twoheadrightarrow} N$. By the induction hypothesis $C[M] \stackrel{\mathbf{R}}{\twoheadrightarrow} C[L]$ and $C[L] \stackrel{\mathbf{R}}{\twoheadrightarrow} C[N]$. Therefore $C[M] \stackrel{\mathbf{R}}{\twoheadrightarrow} C[L]$.

□

Definition 4.16 1. *An* R-redex *is a term M such that $(M, N) \in \mathbf{R}$ for some term N. In this case N is called an* R-contractum *of M.*

2. *A term M is called an* R-normal form *(*R-nf*) if M does not contain (as subterm) any R-redex.*
3. *A term N is an* R-nf of *M (or M* has the R-nf *N) if N is an R-nf and $M \overset{\mathbf{R}}{=} N$.*

Lemma 4.17

$$M \overset{\mathbf{R}}{\to} N \quad \Leftrightarrow \quad M \equiv C[P] \wedge N \equiv C[Q] \wedge (P, Q) \in \mathbf{R}$$
$$\textit{for some } P, Q \in \Lambda, C[\ ] \textit{ with one hole}$$

Proof By definition of $\overset{\mathbf{R}}{\to}$. □

Corollary 4.18 *Let M be an R-nf. Then*

1. *For no N does $M \overset{\mathbf{R}}{\to} N$.*
2. *$M \overset{\mathbf{R}}{\twoheadrightarrow} N \Rightarrow M = N$.*

Proof

1. Immediately from lemma and definition of R-nf.
2. By 1, since $\overset{\mathbf{R}}{\twoheadrightarrow}$ is the reflexive transitive closure of $\overset{\mathbf{R}}{\to}$.

□

Definition 4.19 (Confluence and Diamond Property) 1. *Let $\mathbf{R}$ be a binary relation on Λ. Then $\mathbf{R}$ satisfies the* diamond property *($\mathbf{R} \models \diamond$) if*

$$\forall M, M_1, M_2, (M, M_1) \in \mathbf{R} \wedge (M, M_2) \in \mathbf{R}$$
$$\Rightarrow \exists M_3 : (M_1, M_3) \in \mathbf{R} \wedge (M_2, M_3) \in \mathbf{R}$$

2. *A notion of reduction $\mathbf{R}$ is said to be* confluent *if $\overset{\mathbf{R}}{\twoheadrightarrow}$ satisfies the diamond property.*

5. Combinatory Logic

Definition 5.1 (Generator and Basis)

1. *Let* $\mathfrak{X} \subset A$. *The set of terms* generated by $\mathfrak{X}$, *notation* $\mathfrak{X}^+$, *is the least set* $\mathfrak{Y}$ *such that*
   (a) $\mathfrak{X} \subseteq \mathfrak{Y}$
   (b) $M, N \in \mathfrak{Y} \Rightarrow MN \in \mathfrak{Y}$.
2. *Let* $\mathfrak{A} \subset \Lambda$. $\mathfrak{X} \subset \Lambda$ *is a* basis for $\mathfrak{A}$ *if*

$$\forall M \in \mathfrak{A}, \exists N \in \mathfrak{X}^+ : N = M.$$

3. $\mathfrak{X}$ *is called a* basis *if* $\mathfrak{X}$ *is a basis for* Λ^0.

Proposition 5.2 (K, S Basis) $\{\mathbf{K}, \mathbf{S}\}$ *is a basis. In fact*

$$\forall M \in \Lambda^0, \exists N \in \{\mathbf{K}, \mathbf{S}\}^+ : N \twoheadrightarrow M.$$

6. Representing Data

Definition 6.1 *$F^n(M)$ with $F, M \in \Lambda$ and $n \in \mathbb{N}$ is defined inductively:*

$$\begin{array}{rcl} F^0(M) & = & M \\ F^{n+1}(M) & = & F(F^n(M)) \end{array}$$

Definition 6.2 (Church Numerals) *The Church numerals $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_n$ are defined by*

$$\mathbf{c}_n := \lambda f\, x.f^n(x)$$

Lemma 6.3 1. $(\mathbf{c}_n x)^m(y) = x^{n*m}(y)$

2. $(\mathbf{c}_n)^m(x) = \mathbf{c}_{(n^m)},\ \ m > 0$

Proof

1. Induction on m. $m = 0 \Rightarrow$ LHS $= y =$ RHS. Assume 1 is correct for m (Induction Hypothesis, IH). Then

$$\begin{array}{rcl} (\mathbf{c}_n x)^{m+1}(y) & = & \mathbf{c}_n x((\mathbf{c}_n x)^m(y)) \\ & = & \mathbf{c}_n x(x^{n*m}(y)) \text{ by IH} \\ & = & x^n(x^{n*m}(y)) \\ & = & x^{n+n*m}(y) \\ & = & x^{n*(m+1)}(y). \end{array}$$

2. Induction on $m > 0$. $m = 1 \Rightarrow$ LHS $= \mathbf{c}_n x =$ RHS. Assume 2 is correct for m (Induction Hypothesis, IH). Then

$$\begin{array}{rcl} (\mathbf{c}_n)^{m+1}(x) & = & \mathbf{c}_n((\mathbf{c}_n)^m(x)) \\ & = & \mathbf{c}_n(\mathbf{c}_{(n^m)}(x)) \text{ by IH} \\ & = & \lambda y.(\mathbf{c}_{(n^m)}(x))^n(y) \\ & = & \lambda y.x^{n^m*m}(y)\text{by 1} \\ & = & \mathbf{c}_{(n^{m+1})}x. \end{array}$$

□

Proposition 6.4 (J. B. Rosser) *Define*

$$\begin{array}{rcl} \mathbf{A}_+ & := & \lambda x\, y\, p\, q.x\, p(y\, p\, q) \\ \mathbf{A}_* & := & \lambda x\, y\, z.x(y\, z) \\ \mathbf{A}_{exp} & := & \lambda x\, y.y\, x \end{array}$$

Then $\forall m, n \in \mathbb{N}$

1. $\mathbf{A}_+\mathbf{c}_m\mathbf{c}_n = \mathbf{c}_{m+n}$
2. $\mathbf{A}_*\mathbf{c}_m\mathbf{c}_n = \mathbf{c}_{m*n}$
3. $\mathbf{A}_{exp}\mathbf{c}_m\mathbf{c}_n = \mathbf{c}_{(m^n)},\ m \neq 0$

Proof Exercise. □

Logic

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